A WEAK CASE OF ROTA'S BASIS CONJECTURE FOR ODD DIMENSIONS

DANIEL KOTLAR

Abstract. The Alon-Tarsi Latin square conjecture is extended to odd dimensions by stating it for reduced (normalized) Latin squares. A modified version of Onn's colorful determinantal identity is used to show how the validity of this conjecture implies a weak version of Rota's basis conjecture for odd dimensions, namely, that under a certain condition, the union of n bases can be partitioned into n transversals containing at least $n-1$ bases.

1. Rota's basis conjecture and some conjectures on Latin squares

The following conjecture was stated by G.-C. Rota in 1989 [10]:

Conjecture 1.1 (**Rota's basis conjecture**). *Let $B_1, B_2, \dots, B_n$ be bases of a vector space over an arbitrary field, then $\bigcup_{i=1}^{n} B_i$ can be partitioned to n transversals, each of size n, that are all bases.*

This conjecture can also be stated in terms of matroids, although this paper deals with its slightly narrower version, stated above, for vector spaces. The conjecture is still open, although partial results were solved in [1, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15]. It is well known that this conjecture is related to Latin squares. Let $\mathcal{L}$ be the set of all Latin squares of size $n \times n$ over $\{1, \dots, n\}$. For a latin square $L \in \mathcal{L}$, we use the notations $L_{i,j}$ for its (i,j)th entry, L_i for its ith row and L^j for its jth column. the *sign*, or *parity*, of L, denoted $\operatorname{sgn}(L)$, is defined as the product of the signs of all its row and column permutations, that is, $\operatorname{sgn}(L) = \prod_{i=1}^{n} \operatorname{sgn}(L_i)\operatorname{sgn}(L^i)$. For a given dimension n, $l(n)$ denotes the number of even Latin squares of order n minus the number of odd ones. For odd n it is easy to see that $l(n) = 0$. For even n and for a field of characteristic 0 Conjecture 1.1 was shown in [10] and [13] to be a consequence of the following conjecture of Alon and Tarsi [2]:

Conjecture 1.2 (**Alon-Tarsi Latin square conjecture**). *For all even n, $l(n) \neq 0$.*

No general result for Conjecture 1.1 for odd n has been presented yet. In this paper an analogue of the relation between Conjectures 1.2 and 1.1 is shown for odd n.
The following terms and notations appear in [11] and [16], among others. A Latin square is said to be *normalized* or *reduced* if its first row and first column are the identity permutation. A Latin square is said to be *semi-normalized* if its first column is the identity permutation. A Latin square is said to be *diagonal* if its

Date: September 18, 2011.
1991 *Mathematics Subject Classification.* 68R05, 05B15, 05B20, 11C20, 15A15, 15A03, 52B40.
Key words and phrases. Rota's basis conjecture, Alon-Tarsi conjecture, Latin square, Parity of Latin square.
I thank Ran Ziv for many helpful discussions.

diagonal consists solely of 1's. Let $nels(n)$, $nols(n)$, $sndels(n)$ and $sndols(n)$ be the numbers of normalized even Latin squares, normalized odd Latin squares, semi-normalized diagonal even Latin squares and semi-normalized diagonal odd Latin squares, respectively. Zappa [16] introduced $AT(n) = sndels(n) - sndols(n)$ and proposed the following extension to Conjecture 1.2:

Conjecture 1.3 (**Extended Alon-Tarsi conjecture**). *$AT(n) \neq 0$ for every positive n.*

For even n this conjecture is equivalent to Conjecture 1.2. For odd n, Drisko [6] proved the conjecture in the case that n is prime. It was shown in [16] that $AT(n) = nels(n) - nols(n)$ for even n, but this is not necessarily the case for odd n.

We propose another extension to Conjecture 1.2:

Conjecture 1.4. *$l_r(n) \neq 0$ for every positive n.*

where,

Notation 1.5. *Let $l_r(n) = nels(n) - nols(n)$*

(r for *reduces*).

For even n, Conjecture 1.4 is equivalent to conjectures 1.2 and 1.3, but this is not true for odd n and the conjecture is only known to hold up to $n = 7$ (see [16]). We shall see in Section 3, Theorem 3.3 that the assumption that $l_r(n) \neq 0$ for odd n yields a weak case of Conjecture 1.1, namely the possibility of partitioning into n transversals, of which at least $n-1$ are bases.

2. A modified version of Onn's colorful determinantal identity

The following identity is due to Onn [13]:

Proposition 2.1 (**Onn's colorful determinantal identity**). *Let ${}^1W, {}^2W, \ldots, {}^nW$ be n square matrices of order n over a field F. Then*

$$\sum_{\rho \in S^n} \operatorname{sgn}(\rho) \prod_{i=1}^{n} \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)}\right) = l(n) \prod_{j=1}^{n} \det\left({}^jW\right) \tag{2.1}$$

Where ${}^jW^i$ is the ith column of the matrix ${}^jW$, S^n is the set of n-tuples over the symmetric group S_n and for each such n-tuple $\rho = (\rho_1, \ldots, \rho_n)$, $\operatorname{sgn}(\rho) = \prod_{i=1}^{n} \operatorname{sgn}(\rho_i)$.

The argument in [13] goes as follows: Suppose n is even and Conjecture 1.2 holds. If the columns of each of the matrices ${}^1W, {}^2W, \ldots, {}^nW$ form a base and $\operatorname{char}(F) \nmid l(n)$ then the right hand side of (2.1) is nonzero and thus some term in the sum on the left hand side of (2.1) must be nonzero. Hence, there exists a colorful repartition of the multiset of column of the matrices ${}^iW$ consisting of bases. This implies Conjecture 1.1 for a field of characteristic not dividing $l(n)$. For odd n we know that $l(n) = 0$ and thus we cannot conclude Rota's Conjecture 1.1. In fact, for odd n, the sum on the left hand side of (2.1) can be seen to be zero by a direct

argument: if n is odd, then for any $\pi \in S_n$ and $(\rho_1, \ldots, \rho_n) \in S^n$ we have

$$\prod_{i=1}^{n} \det\left({}^1W^{\pi\rho_1(i)}, \ldots, {}^nW^{\pi\rho_n(i)}\right) = \operatorname{sgn}(\pi)^n \prod_{i=1}^{n} \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)}\right)$$
$$= \operatorname{sgn}(\pi) \prod_{i=1}^{n} \det\left({}^1W^{\rho_1(i)}, \ldots, {}^nW^{\rho_n(i)}\right)$$

It follows that each term in the sum on the left hand side of (2.1) appears $n!/2$ times with a positive sign and $n!/2$ times with a negative sign, and thus the sum on the left hand side of (2.1) is 0.

We shall modify the identity (2.1) so that the left hand side does not contain multiple terms. For this we take only elements of S^n whose first component is the identity permutation. In this case the expression on the right hand side of (2.1) is divided by $n!$:

Proposition 2.2 (**Modified colorful determinantal identity**). *Let* ${}^1W, {}^2W, \ldots, {}^nW$ *be* n *square matrices of order* n *over a field. Then*

$$\sum_{\substack{\rho \in S^n \\ \rho_1 = \mathrm{id}}} \operatorname{sgn}(\rho) \prod_{i=1}^{n} \det\left({}^1W^{i}, {}^2W^{\rho_2(i)}, \ldots, {}^nW^{\rho_n(i)}\right) = \frac{l(n)}{n!} \prod_{j=1}^{n} \det\left({}^jW\right) \tag{2.2}$$

We see that if n is odd we still have zero on the right hand side and we still cannot conclude anything about Conjecture 1.1. However, Proposition 2.2 is a first step in constructing an identity that does no vanish. we shall see in Section 3 how Equation (2.2) can be modified so that the expression on the right hand side of (2.2) becomes nonzero, thus obtaining a result for Conjecture 1.1 for odd n.

Proof of Proposition 2.2. The proof presented here mimics the proof in [13], except that here $\rho_1 = \mathrm{id}$. Let

$$\triangle = \sum_{\substack{\rho, \sigma \in S^n \\ \rho_1 = \mathrm{id}}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} \tag{2.3}$$

We compute $\triangle$ in two different ways. For $\rho = (\mathrm{id}, \rho_2, \ldots, \rho_n)$ let

$$\triangle^{\rho} = \sum_{\sigma \in S^n} \operatorname{sgn}(\sigma) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)}$$
$$= \prod_{i=1}^{n} \sum_{\sigma_i \in S_n} \operatorname{sgn}(\sigma_i) \prod_{j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)}$$
$$= \prod_{i=1}^{n} \det\left({}^1W^{i}, {}^2W^{\rho_2(i)}, \ldots, {}^nW^{\rho_n(i)}\right)$$

Applying the last equation to (2.3) we have

$$\begin{aligned}
\triangle &= \sum_{\substack{\rho\,\in\, S^n \\ \rho_1\,=\,\mathrm{id}}} \operatorname{sgn}(\rho)\triangle^{\rho} \\
&= \sum_{\substack{\rho\,\in\, S^n \\ \rho_1\,=\,\mathrm{id}}} \operatorname{sgn}(\rho)\prod_{i=1}^{n} \det\left({}^{1}W^{i}, {}^{2}W^{\rho_2(i)}, \ldots, {}^{n}W^{\rho_n(i)}\right)
\end{aligned} \tag{2.4}$$

For $\sigma \in S^n$ let

$$\begin{aligned}
\triangle_\sigma &= \sum_{\substack{\rho\,\in\, S^n \\ \rho_1\,=\,\mathrm{id}}} \operatorname{sgn}(\rho) \prod_{i,j=1}^{n} {}^{j}W^{\rho_j(i)}_{\sigma_i(j)} \\
&= \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\sigma_i(1)}\right) \prod_{j=2}^{n} \sum_{\rho_j \in S_n} \operatorname{sgn}(\rho_j) \prod_{i=1}^{n} {}^{j}W^{\rho_j(i)}_{\sigma_i(j)} \\
&= \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\sigma_i(1)}\right) \prod_{j=2}^{n} \det\left({}^{j}W_{\sigma_1(j)}, {}^{j}W_{\sigma_2(j)}, \ldots, {}^{j}W_{\sigma_n(j)}\right)
\end{aligned}$$

Note that $\triangle_\sigma$ is nonzero only for $\sigma = (\sigma_1, \ldots, \sigma_n)$ satisfying that for each $j = 2, \ldots, n$, the set $\{\sigma_1(j), \ldots, \sigma_n(j)\}$ is equal to the set $\{1, \ldots, n\}$. In this case we must have that the set $\{\sigma_1(1), \ldots, \sigma_n(1)\}$ is also equal to the set $\{1, \ldots, n\}$. For each such σ there exists $\pi_\sigma = (\pi_1, \ldots, \pi_n) \in S^n$ so that $\sigma_i(j) = \pi_j(i)$ for all $i, j = 1, \ldots, n$. We have

$$\begin{aligned}
\triangle_\sigma &= \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\sigma_i(1)}\right) \prod_{j=2}^{n} \det\left({}^{j}W_{\sigma_1(j)}, \ldots, {}^{j}W_{\sigma_n(j)}\right) \\
&= \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) \prod_{j=2}^{n} \det\left({}^{j}W_{\pi_j(1)}, \ldots, {}^{j}W_{\pi_j(n)}\right) \\
&= \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) \prod_{k=2}^{n} \operatorname{sgn}(\pi_k) \prod_{j=2}^{n} \det\left({}^{j}W\right) \\
&= \operatorname{sgn}(\pi_\sigma) \prod_{j=2}^{n} \det\left({}^{j}W\right) \operatorname{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right)
\end{aligned} \tag{2.5}$$

Each σ as in (2.5) defines a Latin square L whose rows and columns are the elements of σ and π_σ respectively. thus $\operatorname{sgn}(L) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi_\sigma)$. Substituting (2.5) into (2.3) we have:

$$\begin{aligned}
\triangle &= \sum_{\sigma} \operatorname{sgn}(\sigma)\triangle_\sigma \\
&= \sum_{\sigma} \operatorname{sgn}(\sigma)\operatorname{sgn}(\pi_\sigma) \prod_{j=2}^{n} \det\left({}^{j}W\right) \operatorname{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) \\
&= \prod_{j=2}^{n} \det\left({}^{j}W\right) \sum_{L \in \mathcal{L}} \operatorname{sgn}(L)\operatorname{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right)
\end{aligned} \tag{2.6}$$

Note that π_1 is the first column of the Latin square L. Now, instead of summing over all Latin squares and considering their first column π_1 we sum over all permutations π_1 and then over all Latin squares for which π_1 is their first column. Applying this change to (2.6) we have

$$\begin{aligned} \triangle &= \prod_{j=2}^{n} \det\left({}^{j}W\right) \sum_{L\in\mathcal{L}} \mathrm{sgn}(L)\mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) \\ &= \prod_{j=2}^{n} \det\left({}^{j}W\right) \sum_{\pi_1\in S_n} \mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) \sum_{\substack{L\,\in\,\mathcal{L}\\ \pi_1\,=\,L^1}} \mathrm{sgn}(L) \end{aligned} \tag{2.7}$$

Note that

$$\sum_{\pi_1\in S_n} \mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)}\right) = \det({}^{1}W)$$

and

$$\sum_{\substack{L\,\in\,\mathcal{L}\\ \pi_1\,=\,L^1}} \mathrm{sgn}(L)$$

is the number of even Latin squares with π_1 as their first column minus the number of odd ones. We claim that this number is equal to $l(n)/n!$ (see Lemma 2.4 part (ii) below). Assuming this, Equation (2.7) becomes

$$\triangle = \prod_{j=2}^{n} \det\left({}^{j}W\right) \frac{l(n)}{n!} \det\left({}^{1}W\right) = \frac{l(n)}{n!} \prod_{j=1}^{n} \det\left({}^{j}W\right) \tag{2.8}$$

Combining (2.4) and (2.8) the result follows. □

To conclude this section we need a lemma. First some notation:

Notation 2.3. *Let $\sigma, \pi \in S_n$, $i, j \in \{1, \dots, n\}$*

(i) *$l_{\sigma,i}(n)$ will denote the difference between the numbers of even and odd Latin squares with σ as their ith row.*

(ii) *$l^{\pi,i}(n)$ will denote the difference between the numbers of even and odd Latin squares with π as their ith column.*

(iii) *$l^{\pi,j}_{\sigma,i}(n)$ will denote the difference between the numbers of even and odd Latin squares with σ as their ith row and π as their jth column.*

Lemma 2.4. *Let n be positive, $\pi \in S_n$, $1 \le i \le n$*

(i) *If $n \ge 3$ is odd then $l_{\pi,i}(n) = l^{\pi,i}(n) = 0$*

(ii) *For all n, $l_{\pi,1}(n) = l^{\pi,1}(n) = l(n)/n!$*

Proof. (i) For a given i, fix some j and k different from i. For any Latin square with π as its ith row we can obtain a Latin square with the opposite parity by exchanging the jth and kth rows. Similarly for columns.
(ii) If n is odd then $l_{\pi,1}(n) = l^{\pi,1}(n) = l(n) = 0$ by part (i). Suppose n is even. Let $\pi_1, \pi_2 \in S_n$. Let L be a Latin square containing π_1 as its ith row. By applying the permutation $\pi_2 \circ \pi_1^{-1}$ on the rows of L we obtain a Latin square with the same parity containing π_2 as its ith row. Thus $l_{\pi_1,1}(n) = l_{\pi_2,1}(n) = l(n)/n!$, and similarly for columns. □

3. A weak case of Rota's basis conjecture for odd n

We saw in Section 2 that the colorful determinantal identity cannot be used to conclude anything about Rota's basis conjecture for odd dimensions. In this section we shall invert the signs of half of the terms in the sum on the left hand side of Equation (2.2) so that the sum on the right hand side will not vanish.
Recall Notations 1.5 and 2.3.

Lemma 3.1. *Let $\sigma, \pi \in S_n$.*

(i) *if n is even then $l_{\sigma,1}^{\pi,1}(n) = l_r(n)$.*
(ii) *if n is odd then $l_{\sigma,1}^{\pi,1}(n) = \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi) l_r(n)$.*

Before proving the lemma recall from [11] that an *isotopy* is a triple (α, β, γ) such that $\alpha, \beta, \gamma \in S_n$ and it acts on a Latin square L by applying α on the set of rows, β on the set of columns and γ on the symbols of the square.

Proof of Lemma 3.1. Let L be a Latin square containing σ as its first row and π as its first column. If $L_{1,1} = k \neq 1$, we apply the inversion $\gamma = (1, k)$ on the set $\{1, \ldots, n\}$ to obtain a square with 1 as its $(1,1)$ entry (If $L_{1,1} = 1$ then $\gamma = \mathrm{id}$). Now apply a permutation α on the rows and a permutation β on the columns to obtain a reduced (normalized) Latin square. Since α and β are determined by π and σ respectively, the isotopy (α, β, γ) can be applied on any square containing σ as its first row and π as its first column, to obtain a reduced square. Now α and β have the same parity if and only if σ and π have the same parity (since $\alpha \circ \gamma = \pi^{-1}$ and $\beta \circ \gamma = \sigma^{-1}$). According to Proposition 3.1 in [11] the resulting square and the original square have opposite parities only in the case that n is odd and $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\pi)$. $\square$

Theorem 3.2. *Let ${}^1W, {}^2W, \ldots, {}^nW$ be n square matrices of order n over a field. Suppose n is odd and Conjecture 1.4 holds for n, then*

$$\sum_{\substack{\rho \in S^n \\ \rho_1 = \mathrm{id}}} \mathrm{sgn}(\rho)\mathrm{perm}\left({}^1W^1, {}^2W^{\rho_2(1)}, \ldots, {}^nW^{\rho_n(1)}\right) \prod_{i=2}^{n} \det\left({}^1W^i, {}^2W^{\rho_2(i)}, \ldots, {}^nW^{\rho_n(i)}\right) = (n-1)! \cdot l_r(n)\mathrm{perm}\left({}^1W\right) \prod_{j=2}^{n} \det\left({}^jW\right) \tag{3.1}$$

Proof. On the right hand side of (2.2) we have a sum of $(n!)^{n-1}$ terms each consisting of the product of n determinants. If we omit the signs in the first determinant of each such term we get the product of $n-1$ determinants and one permanent. We shall see that this can be achieved by omitting $\mathrm{sgn}(\sigma_1)$ in (2.3). We denote the resulting expression by $\triangle'$ instead of $\triangle$ and Equation (2.3) takes the following form:

$$\begin{aligned} \triangle' &= \sum_{\substack{\rho, \sigma \in S^n \\ \rho_1 = \mathrm{id}}} \frac{\mathrm{sgn}(\sigma)}{\mathrm{sgn}(\sigma_1)} \mathrm{sgn}(\rho) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} \\ &= \sum_{\substack{\rho, \sigma \in S^n \\ \rho_1 = \mathrm{id}}} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma)\mathrm{sgn}(\rho) \prod_{i,j=1}^{n} {}^jW_{\sigma_i(j)}^{\rho_j(i)} \end{aligned}$$

We can compute $\triangle'$ in two different ways. Taking the external sum by ρ we obtain

(3.2)
$$\begin{aligned}
\triangle' &= \sum_{\substack{\rho\,\in\, S^n\\ \rho_1\,=\,\mathrm{id}}} \mathrm{sgn}(\rho) \sum_{\sigma\in S^n} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma) \prod_{i,j=1}^{n} {}^{j}W^{\rho_j(i)}_{\sigma_i(j)} \\
&= \sum_{\substack{\rho\,\in\, S^n\\ \rho_1\,=\,\mathrm{id}}} \mathrm{sgn}(\rho) \left(\sum_{\sigma_1\in S_n} \prod_{j=1}^{n} {}^{j}W^{\rho_j(1)}_{\sigma_1(j)} \right) \left(\prod_{i=2}^{n} \sum_{\sigma_i\in S_n} \mathrm{sgn}(\sigma_i) \prod_{j=1}^{n} {}^{j}W^{\rho_j(i)}_{\sigma_i(j)} \right) \\
&= \sum_{\substack{\rho\,\in\, S^n\\ \rho_1\,=\,\mathrm{id}}} \mathrm{sgn}(\rho) \mathrm{perm}\left({}^{1}W^{1}, {}^{2}W^{\rho_2(1)}, \dots, {}^{n}W^{\rho_n(1)}\right) \prod_{i=2}^{n} \det\left({}^{1}W^{i}, {}^{2}W^{\rho_2(i)}, \dots, {}^{n}W^{\rho_n(i)}\right)
\end{aligned}$$

Taking the external sum by σ and substituting $\triangle_\sigma$ from (2.5) we obtain

(3.3)
$$\begin{aligned}
\triangle' &= \sum_{\sigma} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma)\triangle_\sigma \\
&= \sum_{\sigma} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma)\mathrm{sgn}(\pi) \prod_{j=2}^{n} \det\left({}^{j}W\right) \mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)} \right) \\
&= \prod_{j=2}^{n} \det\left({}^{j}W\right) \sum_{L\in\mathcal{L}} \mathrm{sgn}(\sigma_1)\mathrm{sgn}(L)\mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)} \right) \\
&= \prod_{j=2}^{n} \det\left({}^{j}W\right) \sum_{\pi_1\in S_n} \mathrm{sgn}(\pi_1) \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)} \right) \sum_{\substack{\sigma_1\,\in\, S_n\\ \pi_1(1)\,=\,\sigma_1(1)}} \mathrm{sgn}(\sigma_1) \sum_{\substack{L\,\in\,\mathcal{L}\\ \sigma_1\,=\,L_1\\ \pi_1\,=\,L^1}} \mathrm{sgn}(L)
\end{aligned}$$

Now, according to Lemma 3.1, for any $\pi_1, \sigma_1 \in S_n$

$$\mathrm{sgn}(\pi_1)\mathrm{sgn}(\sigma_1) \sum_{\substack{L\,\in\,\mathcal{L}\\ \sigma_1\,=\,L_1\\ \pi_1\,=\,L^1}} \mathrm{sgn}(L) = l_r(n)$$

Also, for each $\pi_1 \in S_n$ there are $(n-1)!$ permutations $\sigma_1 \in S_n$ so that $\pi_1(1) = \sigma_1(1)$. Hence

$$\begin{aligned}
\triangle' &= \prod_{j=2}^{n} \det\left({}^{j}W\right) (n-1)! \cdot l_r(n) \sum_{\pi_1\in S_n} \left(\prod_{i=1}^{n} {}^{1}W^{i}_{\pi_1(i)} \right) \\
&= \prod_{j=2}^{n} \det\left({}^{j}W\right) (n-1)! \cdot l_r(n) \mathrm{perm}\left({}^{1}W\right)
\end{aligned} \tag{3.4}$$

Combining (3.2) and (3.4) the result follows. □

We can now obtain a weak version of Rota's basis conjecture for odd n:

Theorem 3.3. *Suppose n is odd and $l_r(n) \neq 0$. Let $B_1, B_2, \dots, B_n$ be bases of a vector space of dimension n over a field of characteristic 0. Suppose the matrix whose column are the vectors of B_1 has nonzero permanent, then $\bigcup_{i=1}^{n} B_i$ can be*

partitioned into n transversals, each of size n, such that at least $n-1$ of them are bases.

Proof. For each $j = 1, \dots, n$ let ${}^jW$ be the matrix whose columns are the elements of B_j. Then, by the assumptions of the theorem, the right hand side of (3.1) is nonzero. It follows that at least one of the terms in the sum on the left hand side of (3.1) is nonzero. This term gives $n-1$ transversals that are bases. □

Remark: For the result of Theorem 3.3 to hold B_1 need not be a base. However, the theorem was stated in this way for simplicity and in order to keep the hypotheses of the theorem as similar as possible to those of Conjecture 1.1.

References

1. R. Aharoni and E. Berger, *The intersection of a matroid and a simplicial complex*, Trans. Amer. Math. Soc. **358** (2006), 4895–4917.
2. N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), no. 2, 125–134.
3. W. Chan, *An exchange property of matroid*, Discrete Math. **146** (1995), 299–302.
4. T. Chow, *On the Dinitz conjecture and related conjectures*, Discrete Math. **145** (1995), 73–82.
5. A. A. Drisko, *On the number of even and odd Latin squares of order $p+1$*, Advances in Math. **128** (1997), 20–35.
6. A. A. Drisko, *Proof of the Alon-Tarsi conjecture for $n = 2^r p$*, The Electronic Journal of Combinatorics (1998), no. R28.
7. J. Geelen and P. J. Humphries, *Rotas basis conjecture for paving matroids*, SIAM J. Discrete Math. **20** (2006), 1042–1045.
8. J. Geelen and K. Webb, *On Rotas basis conjecture*, SIAM J. Discrete Math. **21** (2007), 802–804.
9. D. Glynn, *The conjectures of Alon-Tarsi and Rota in dimension prime minus one*, SIAM J. Discrete Math. **24** (2010), 394–399.
10. R. Huang and G-C. Rota, *On the relations of various conjectures on Latin squares and straightening coefficients*, Discrete Mathematics **128** (1994), 225–236.
11. J. C. M. Janssen, *On even and odd Latin squares*, Journal of Combinatorial Theory A **69** (1995), 173–181.
12. D. Kotlar and R. Ziv, *On partitioning two matroids into common independent subsets*, Discrete Mathematics **300** (2005), 239–244.
13. S. Onn, *A colorful determinantal identity, a conjecture of Rota, and Latin squares*, The American Mathematical Monthly **104** (1997), no. 2, 156–159.
14. V. Ponomarenko, *Reduction of jump systems*, Houston J. Math. **30** (2004), 27–33.
15. M. Wild, *On Rotas problem about n bases in a rank n matroid,*, Advances in Math. **108** (1994), 336–345.
16. P. Zappa, *The Cayley determinant of the determinant tensor and the Alon-Tarsi conjecture*, Advances in Applied Mathematics **19** (1997), 31–44.

Computer Science Department, Tel-Hai College, Upper Galilee 12210, Israel
E-mail address: dannykot@telhai.ac.il